Gravitational potential energy of simple bodies: the homogeneous bispherical concavo-convex lens

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Gravitational potential energy of a homogeneous bispherical concavo-convex lens, with the equal absolute values of curvature radii of both surfaces, is studied. Compactness factor notion is introduced and calculated as function of central thickness of lens.

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Introduction

At present, the potential energy is known analytically for three types of *homogeneous* self-gravitating bodies: a) ellipsoids [1], b) concave bispherical lenses [2], and 3) rectangular parallelepipeds [3]. In accompaning paper [4], we present a method of negative density which allows to obtain the analytical solutions for potential energy of new kinds of "simple" self-gravitating bodies. Here we present the potential energy for the homogeneous bispherical concavo-convex lens with radii of curvature $R_2 = -R_1$. Such figure is obtained by cutting out a symmetric bispheric lens from the sphere (Fig. 1). If we introduce H, a central thickness of lens, then potential energy of the "quasi-symmetric" bispheric lens is:

$$W = \frac{\pi}{540\,H}\, G\,\rho^2\, R^5 \left[40\,a\,H\,\left(-12 - 16\,H^2 + H^4\right) + \right. \\ \left. 3\,\pi\,\left(160 - 240\,H^2 + 30\,H^4 - H^6\right) + 960\,\left(-2 + 3\,H^2\right)\,\arctan \frac{2-H}{2\,a} \right]; \tag{1}$$

here G is constant of Newtonian gravitation, $\rho = constant$ is a matter density, R is radius of spherical surfaces, H is central thickness (in units of R), and $a = \sqrt{1 - H^2/4}$. (Note that in the caption to Fig. 1 all values are dimensional.) We call the figure in question "quasi-symmetrical", as in general case, two spherical surfaces may have the different absolute values of curvature radii. This more general case will be considered elsewhere.

Dependence of potential energy of the lens on central thickness

The series expansions of W at points $H = 0$ and $H = 2$ are:

$$\begin{aligned} W_s(0)/(G\,\rho^2\,R^5) &= \frac{-64\,H^2\,\pi}{27} + \frac{8\,H^4\,\pi}{45} + \frac{H^3\,\pi^2}{6}; \\ W_s(2)/(G\,\rho^2\,R^5) &= \frac{-16\,\pi^2}{15} + \frac{2\,(-2+H)^2\,\pi^2}{3}. \end{aligned} \tag{2}$$

In general, the function $W(H)$ is monotonic (Fig. 2), simply because the volume and mass (and so potential energy) of the lens are all increasing with increasing H.

Compactness factor

It is of some interest to consider "compactness" of the lens as function of H. We introduce compactness factor, as characteristic of the potential energy of the honogeneous bodies, as the dimensionless coefficient :

$$w = -\frac{W}{G\,\rho^2 V^{5/3}}; \tag{3}$$

in general case of inhomogeneous bodies, it is better to use mean density in the definition of w. As an example, for the honogeneous sphere we have:

$$w_{sphere} = \frac{3}{5}(\frac{4\,\pi}{3})^{1/3} = .967195. \tag{4}$$

This is the maximal possible value of compactness factor for homogeneous bodies.
The volume of the lens is:

$$V = \left(H - \frac{H^3}{12}\right)\,\pi\,R^3. \tag{5}$$

The dependence of compactness factor w for quasi-symmetric concavo-convex lens on central thickness H is shown in Fig. 3. The serial expansion of w at point $H = 0$ is:

$$w_s(0) = \frac{H^{\frac{1}{3}}\,(128 - 9\,H\,\pi)}{54\,\pi^{\frac{2}{3}}}, \tag{6}$$

which is also shown in Fig. 3.

Conclusion

In conclusion, we present here the analytic formula for gravitational energy of the homogeneous bispheric concavo-convex lens, when the radii of curvature of both surfaces of the lens have the same absolute value. Both potential energy and compactness factor are shown to be monotonic increasing functions of relative central thickness of the lens. As to applicational aspect of the concavo-convex lens problem, we may mention the modelling of the solar system's small bodies with large craters, such as Phobos with his large Stickney crater.

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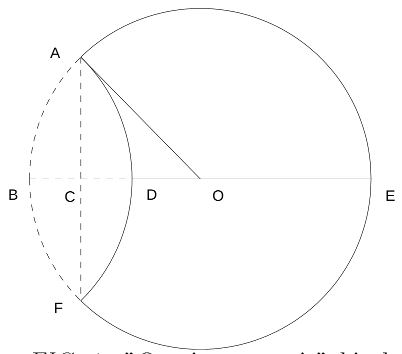

FIG. 1. "Quasi-symmetric" bispherical concavo-convex lens. Radius of sphere is $R = OA = OB = OE$; central thickness is $H = DE$; radius, a, of base of additional symmetric bispheric lens ABFD is $a = AC = CF$; half-height, h, or central half-thickness of lens ABFD is $h = (2\,R - H)/2 = R - H/2$; and $a = \sqrt{R^2 - (H + h - R)^2} = \sqrt{R^2 - H^2/4}$.

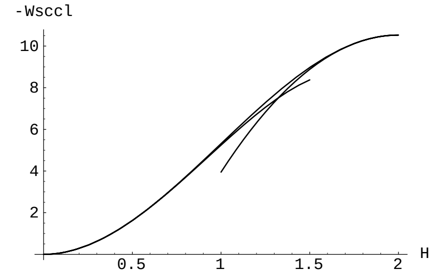

FIG. 2. Gravitational potential energy, $-W/(G\,\rho_2\,R^5)$, of "quasi-symmetric" bispherical concavo-convex lens as function of dimensionless central thickness H. Note that limit $H->0$ corresponds to "new-born-Moon"-like crescent and limit $H->2$ corresponds to a full sphere with $-W/(G\,\rho_2\,R^5) = 16/15\,\pi^2$. Also shown are series expansions according to Eq. (2).

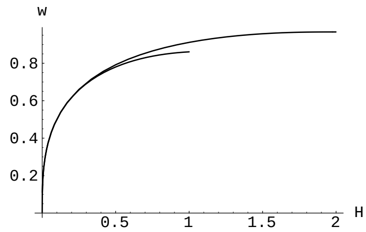

FIG. 3. Compactness factor of "quasi-symmetric" bispherical concavo-convex lens as function of dimensionless central thickness H, according to Eqs. (1), (3) and (5).Also shown is the series expansion according to Eq. (6).